Experimental and numerical validation of Advanced Statistical Energy Analysis to incorporate tunneling mechanisms for vibration transmission across a grillage of beams

Xing Wang and Carl Hopkins

Acoustics Research Unit, School of Architecture, Abercromby Square, University of Liverpool, Liverpool L69 7ZN, United Kingdom

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Abstract — Advanced Statistical Energy Analysis (ASEA) is used to predict vibrational response on a three-bay linear grillage of beams that supports multiple wave types when there is significant indirect coupling through tunneling mechanisms. For bending wave excitation where the component beams have identical material properties, there was agreement between measurements, ASEA and FEM (Finite Element Methods). The importance of indirect coupling was confirmed for bending-longitudinal and bending-torsional models due to ASEA predicting a higher response than SEA on beams that were distant from the source, and closer agreement between FEM and ASEA (rather than SEA) with only bending modes on all the beams or where beams supported longitudinal or torsional modes as well as bending modes. To investigate an imperfectly periodic, finite grillage that could exist due to engineering tolerances, numerical experiments with FEM were used to introduce uncertainty into the Young’s modulus for each beam. For beams modelled with Euler-Bernoulli or Timoshenko theory, the effect of this uncertainty was to reduce differences between FEM and ASEA to less than 3 dB. The results confirm the ability of ASEA to predict vibration transmission with significant indirect coupling across frameworks of beams that support local modes with multiple wave types.

Keywords: Beam, Tunneling mechanisms, Indirect coupling, Periodic structure, Uncertainty

1 Introduction

Large frameworks of beams in engineering structures are often constructed from repeating units where each end of the beam is connected to at least two other beams, such as in a truss beam or grillage. These spatially periodic structures occur in buildings, aircraft, marine craft and space structures for which it is necessary to predict vibration transmission for the purpose of noise control or to ensure low vibrational activity when mounting sensitive equipment.

Previous work by the authors [1] used Advanced Statistical Energy Analysis (ASEA) [2] to model a rectangular beam framework which was shown to be able to predict the high propagation losses at high frequencies. This paper extends the assessment of ASEA to determine whether vibration transmission can be predicted across a finite, periodic structure where there is significant indirect coupling, i.e. tunnelling mechanisms. A three-bay linear grillage of beams is considered through comparison of ASEA with measurements, Finite Element Methods (FEM), and Statistical Energy Analysis (SEA).

Deterministic approaches are often used to predict vibration transmission across networks of beams, particularly for periodic frames that can act as a mechanical filter showing pass and stop bands. Signorelli and von Flotow [3] used a transfer matrix approach to model bending, shear and longitudinal wave propagation across a periodic truss beam formed from rods with pinned joints. Over a frequency range spanning the first five modes of each bay this showed pass and stop band features with complex modes. Miller and von Flotow [4] used a wave scattering approach to determine power flow across a network of one-dimensional elements. For application to a wider range of beam networks, Langley [5] considered the direct-dynamic stiffness method and showed that this approach is suited to high-frequency power flow analysis. Later work by Shorter and Langley [6] introduced an approach to vibroacoustics modelling that was highly flexible and could combine deterministic and statistical descriptions of subsystems to deal with the “mid-frequency problem” as well as indirect coupling. Shankar and Keane [7] used a general method based on receptance theory to evaluate power flow in Euler-Bernoulli beam frames with rigid joints. This was validated against finite element models for uniform internal damping on all beams. It was then used to assess a network of beams with different damping on individual beams to investigate...
the potential to reduce the vibratory response. Beale and Accorsi [8] generalized the wave scattering approach of Miller and von Flotow for complex frames with multiple wave types. They considered bending, longitudinal and torsional waves on three-dimensional beam frames using Timoshenko beam theory because the approach was intended for mid- and high-frequencies. Each beam member was treated as a wave guide, so that only the power at the junctions was evaluated. Numerical results were shown to be accurate and computationally efficient compared with FEM. For a two-dimensional grillage of beams excited with a harmonic point load, Langley et al. [9] compared predictions and measurements for bending and torsional wave transmission. This confirmed the existence of spatial filtering properties and frequency-dependant, directional ‘beaming’ behaviour. To give insights into coupled bending and longitudinal wave motion, Yun and Mak [10] used the transfer matrix method on a semi-infinite, periodic framework of beams.

In terms of statistical approaches to predicting vibration transmission, Sablik [11] considered how discrete structural resonances for frameworks of beams could be incorporated in SEA with bending, longitudinal and torsional waves by modifying the coupling using mode counts for each beam. An irregular network of concrete beams representing a building structure was modelled, but comparisons with experimental work produced unrealistic level differences which did not correspond to physical reality. This was partly because results were shown up to 40 kHz without any consideration of bending theory for thick beams in the SEA model. Cho and Bernhard [12] used Energy Flow Analysis (EFA) on a three-dimensional beam junction comprised of three beams to predict vibration transmission due to bending in two perpendicular directions, as well as longitudinal and torsional waves. EFA was used to predict the frequency-average response but its advantage over SEA is that it also predicts spatial variation in the response along the beam. EFA was compared against the exact response of a three-dimensional beam junction; this indicated that the agreement was typically better than 5 dB. For a network of 12 beams, You et al. [13] compared SEA with EFA that was modified for broadband excitation, which indicated they were similar for beams that were directly coupled to the excited beam but not for beams that were distant from the excited beam.

In engineering structures, grillages of beams can be formed from L-, T- or X-junctions of coupled beams. This paper considers a three-bay linear grillage of beams for which the component junctions are L- and T-junctions. The transmission coefficients are determined using wave theory based on semi-infinite Euler-Bernoulli beams for a bending and longitudinal wave model (BL model), and a bending and torsional wave model (BT model). The wave theory for L-junctions is given in previous work [1]; hence, only the derivations for T-junctions are given in the appendices of this paper.

For a T-junction, Figure 1 shows the two cases that need consideration for excitation on beam 1 of the T123-junction and the cantilever beam (beam 1) of a T124-junction. Assuming only bending (Euler-Bernoulli theory) and longitudinal waves on semi-infinite beams, Cremer et al. [14] stated the bending wave transmission coefficients for T123-junctions. In a later edition of the book, Cremer et al. [15] gave a general derivation for an X-junction where all beams could have different material properties and different cross-sections. This approach was then used to derive results for a T-junction. However, for these T- and X-junctions, only asymptotic expressions were given for bending wave transmission coefficients (i.e. not those involving longitudinal wave motion) and the graphs of the transmission coefficients gave no indication that some values can be frequency-dependent. Horner and White [16] derived transmission coefficients for three beams coupled together at a single junction which had variable angles between them and assumed that the joint acted as a rigid mass; hence these beams could also form a T-junction (it appears that there is a typographical error in equation (15) in [16] where the last term should be a second-order time derivative instead of a first-order space derivative). For bending (Euler-Bernoulli theory) and torsional waves on semi-infinite beams, Heckl [17] derived wave theory transmission coefficients for a T123-junction considering bending and torsional wave excitation, respectively. Moore [18] and Tso and Norwood [19] introduced general frameworks to calculate transmission coefficients for two- and three-dimensional junctions of beams respectively. Tso and Norwood’s framework also allowed for junctions where there was an offset in the beam centroid and shear centre. For the same T-junction of I-beams, Moore showed that there tends to be weak transmission of torsional and in-plane bending waves and Tso and Norwood showed that the inclusion of shear deformation and rotatory inertia at high frequencies affected high-frequency transmission of bending and torsional waves.

In this paper, the derivations for a T-junction with a rigid massless joint have been re-visited in order to (a) produce a set of consistent derivations with excitation by bending, longitudinal and torsional waves by allowing different materials and cross-section to be used, (b) provide derivations for the T-junction of the bending and longitudinal wave model which were not considered in [14] and (c) derive transmission coefficients with the T124-junction for the bending and torsional model wave which were not considered in reference [17].
ASEA is a high-frequency model which combines SEA and ray tracing in order to track the power transmitted between coupled subsystems [2]. Phase effects are not considered in ASEA; hence there is no reason to assume a priori that it would be suitable for all periodic frameworks of beams. However, Yin and Hopkins [20] have experimentally validated the use of ASEA to model bending wave transmission across a plate junction which incorporates a periodic ribbed plate. This confirmed that when each bay on the ribbed plate supported local modes, ASEA was able to account for significant indirect coupling between non-adjacent bays. Close agreement between ASEA and measurements indicated that the absence of phase information was not critical.

This paper considers whether ASEA incorporating multiple wave types can predict vibration transmission across a finite, periodic framework of beams, when this framework is (a) “perfectly periodic” (identical material properties and geometry) and (b) “imperfectly periodic” with uncertainty in the Young’s modulus of the beams. In previous work [1], seven identical L-junctions were built from Perspex beams and tested. These gave nominally identical results in terms of the modal fluctuations. Therefore, by using the same material and type of connection at the junctions for the three-bay linear grillage, it is reasonable to consider it as a close approximation to “perfectly periodic”. However, introducing variation in the Young’s modulus of the Perspex gives an opportunity to use FEM to consider the effect of uncertainty in the material properties on the response. The form of excitation used to excite bending, longitudinal or torsional wave motion is rain-on-the-roof (i.e. point forces over the length of the beam with unity magnitude and random phase). To reduce the effect of propagation losses and increase the likelihood of tunnelling, the beams are relatively short in length. However, this reduces the mode count in each frequency band for the beam subsystems; hence octave bands are used to overcome this issue. This results in consecutive frequency bands having at least one local mode on the source or receiving beams to facilitate comparison between measurements, FEM, ASEA, and SEA. To carry out analysis across a broad frequency range, Euler-Bernoulli and Timoshenko theory is incorporated into the prediction models.

2 Wave theory transmission coefficients for the T-junction

The SEA and ASEA theories that are used in this paper are fully described in [1]; hence this section focusses on the transmission coefficients that are needed to calculate the coupling loss factors in these models. For the T123- and T124-junctions it is assumed that the material properties and the cross-sectional dimensions are identical for beams 1 and 3, and beams 2 and 4. For a two-dimensional junction, four possible incident waves can be considered: Type A bending waves (with displacement in the same plane as the junction), Type B bending waves (with displacement perpendicular to the plane of the junction), longitudinal waves and torsional waves. For beams that are perpendicular to each other at the junction, excitation of Type A bending waves generates longitudinal waves at the junction, and excitation of Type B bending waves generates torsional waves at the junction.

The BT model for a T123-junction is described by Heckl [17]; hence this section only contains the final matrices used to solve the BL models for the T123- and T124-junctions and the BT model for the T124-junction with full derivations for the transmission coefficients given in Appendix A.

2.1 BL model for the T123-junction

Figure 2 shows the T123-junction under consideration for which the coordinates of the junction line are \((x, y) = (0, 0)\).

For Type A bending wave excitation (Fig. 2a), the boundary conditions described in Appendix A give the matrix equation as:

\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 1 & -1 & -1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 \\
i & 1 & -i & 0 & 0 & 0 & 0 \\
i & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -i & 0 & 0 & 2i & 0 \\
0 & 0 & -i & 0 & 0 & 0 & -i \\
1 & -1 & -\psi & \psi & -1 & 1 & 0 \end{bmatrix}
\begin{bmatrix}
\tau_{B_{11}} \\
\tau_{N_{11}} \\
\tau_{B_{12}} \\
\tau_{N_{12}} \\
\tau_{B_{13}} \\
\tau_{N_{13}} \\
\tau_{B_{14}} \\
\tau_{N_{14}} \\
\tau_{B_{15}} \\
\tau_{N_{15}} \\
\tau_{B_{16}} \\
\tau_{N_{16}} \end{bmatrix} = \begin{bmatrix}
-1 \\
-1 \\
0 \\
i \\
i \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \end{bmatrix}
\]

where \(\chi, \psi, \beta_1\) and \(\beta_2\) are defined as

\[
\chi = \frac{k_{B_{2r}}}{k_{B_{1r}}}, \quad \psi = \frac{B_{3}k_{B_{2r}}}{B_{1}k_{B_{1r}}}, \quad \beta_1 = \frac{m_{2}c_{B_{2}}}{m_{1}c_{L_{1}}}, \quad \beta_2 = \frac{m_{1}c_{B_{1}}}{m_{2}c_{L_{2}}}. \quad (2)
\]

Solving equation (1) allows the reflection and transmission coefficients to be calculated as follows:

\[
\tau_{B_{11}} = |\tau_{B_{11}}|^2, \quad (3)
\]

\[
\tau_{B_{12}} = \chi^2 |\tau_{B_{12}}|^2, \quad (4)
\]

\[
\tau_{B_{13}} = |\tau_{B_{13}}|^2, \quad (5)
\]

\[
\tau_{B_{14}} = \chi^2 |\tau_{B_{14}}|^2, \quad (6)
\]

\[
\tau_{B_{15}} = \frac{1}{2\beta_1} |\tau_{B_{15}}|^2. \quad (7)
\]

For longitudinal wave excitation (Fig. 2b) as described in Appendix A, the boundary conditions generate the following matrix equation:

\[
\begin{bmatrix}
i & 1 & i\chi & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \beta_1 & i\beta_1 & 1 & 1 & 0 \\
2 & -2 & -\psi & \psi & 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix}
\tau_{L_{11}} \\
\tau_{N_{11}} \\
\tau_{L_{12}} \\
\tau_{N_{12}} \\
\tau_{L_{13}} \\
\tau_{N_{13}} \end{bmatrix} = \begin{bmatrix}
0 \\
1 \\
1 \\
0 \\
1 \\
0 \end{bmatrix}. \quad (8)
\]
The solution of equation (8) allows the reflection and transmission coefficients to be calculated as follows:

\[
\tau_{L1B1} = \tau_{L1B3} = \frac{2\beta_1}{\gamma} |l_{L1B1}|^2, \tag{9}
\]

\[
\tau_{L1B2} = 2\beta_1 |l_{L1B2}|^2, \tag{10}
\]

\[
\tau_{L1L1} = |l_{L1L1}|^2 \tag{11}
\]

\[
\tau_{L1L2} = 0, \tag{12}
\]

\[
\tau_{L1L3} = |l_{L1L3}|^2. \tag{13}
\]

### 2.2 BL model for the T124-junction

Figure 3 shows the T124-junction under consideration for which the coordinates of the junction line are \((x, y) = (0, 0)\).

For Type A bending wave excitation (Fig. 3a), the boundary conditions described in Appendix A give the matrix equation as:

\[
\begin{pmatrix}
i & 1 & i\gamma & 0 \\
1 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 1 & 0 \\
\beta_2 & i\beta_2 & 0 & 0 & 2 \\
1 & -1 & -2\psi & 2\psi & 0
\end{pmatrix}
\begin{pmatrix}
r_{B1B1} \\
r_{N1} \\
t_{B1B2} \\
t_{N2} \\
t_{B1L2}
\end{pmatrix}
= \begin{pmatrix}
i \\
-1 \\
0 \\
\beta_2 \\
-1
\end{pmatrix}. \tag{14}
\]

Solving equation (14) allows the reflection and transmission coefficients to be calculated as follows:

\[
\tau_{B1B1} = |r_{B1B1}|^2, \tag{15}
\]

\[
\tau_{B1B2} = \tau_{B1B4} = \gamma\psi |l_{B1B2}|^2. \tag{16}
\]

\[
\tau_{B1L1} = 0, \tag{17}
\]

\[
\tau_{B1L2} = \frac{1}{2\beta_2} |t_{B1L2}|^2. \tag{18}
\]

For longitudinal wave excitation (Fig. 3b) as described in Appendix A, the boundary conditions give the following matrix equation:

\[
\begin{pmatrix}
-1 & 1 & 0 & 1 \\
0 & i & 0 & 1 \\
1 & 2\beta_1 & 2i\beta_1 & i \\
0 & 1 & 0 & 2\mu_1 \\
-1 & 1 & 0 & 0 & 2\mu_2 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
r_{L1L1} \\
r_{L1B2} \\
t_{L1B1} \\
t_{L1B2} \\
t_{L1L2}
\end{pmatrix}
= \begin{pmatrix}
1 \\
0 \\
1 \\
0 \\
1
\end{pmatrix}. \tag{19}
\]

The solution of equation (19) allows the reflection and transmission coefficients to be calculated as follows:

\[
\tau_{L1L1} = |r_{L1L1}|^2, \tag{20}
\]

\[
\tau_{L1L2} = \tau_{L1L4} = 0, \tag{21}
\]

\[
\tau_{L1B1} = 0, \tag{22}
\]

\[
\tau_{L1B2} = \tau_{L1B4} = 2\beta_1 |l_{L1B2}|^2. \tag{23}
\]

### 2.3 BT model for the T124-junction

Figure 4 shows the T124-junction under consideration for which the coordinates of the junction line are \((x, y) = (0, 0)\).

For Type B bending wave excitation (Fig. 4a), the boundary conditions described in Appendix A give the matrix equation as:

\[
\begin{pmatrix}
-1 & -1 & i & 1 \\
1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & i & 1 \\
-1 & 1 & 0 & 0 & 2\mu_1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
r_{B1B1} \\
r_{N1} \\
t_{B1B2} \\
t_{N2} \\
t_{B1T2}
\end{pmatrix}
= \begin{pmatrix}
1 \\
i \\
0 \\
i \\
1
\end{pmatrix}. \tag{24}
\]

where \(\mu_1, \mu_2\) and \(\beta\) are defined as

\[
\mu_1 = \frac{oZ_{T2}B_1k_{B1}}{B_1k_{B1}}, \mu_2 = \frac{oZ_{T1}B_2k_{B2}}{B_1k_{B1}}, \beta = \frac{B_2k_{B2}^3}{B_1k_{B1}}. \tag{25}
\]

Solving equation (24) allows the reflection and transmission coefficients to be calculated as follows:

\[
\tau_{B1B2} = |r_{B1B2}|^2, \tag{26}
\]

\[
\tau_{B1B2} = \tau_{B1B4} = \beta |t_{B1B2}|^2, \tag{27}
\]

\[
\tau_{B1T1} = 0, \tag{28}
\]

\[
\tau_{B1T2} = \tau_{B1T4} = \frac{\mu_1}{2} |t_{B1T2}|^2. \tag{29}
\]
For torsional wave excitation (Fig. 4b) as described in Appendix A, the boundary conditions give the following matrix equation:

\[
\begin{bmatrix}
-1 & i & -1 \\
1 & 1 & 0 \\
-2 & 2 & \mu_2
\end{bmatrix}
\begin{bmatrix}
\tau_{T1B2} \\
\tau_{N2} \\
\rho_{T1T1}
\end{bmatrix}
= \begin{bmatrix}
1 \\
0 \\
\mu_2
\end{bmatrix}.
\] (30)

The solution of equation (30) allows the reflection and transmission coefficients to be calculated as follows:

\[
\tau_{T1T1} = |\rho_{T1T1}|^2, \\
\tau_{T1T2} = \tau_{T1T4} = 0, \\
\tau_{T1B1} = 0, \\
\tau_{T1B2} = \tau_{T1B4} = \frac{2}{\mu_2} |\tau_{T1B2}|^2.
\] (31-34)

3 Experimental work

3.1 Grillage construction

The design of the three-bay linear grillage (see Fig. 5) uses relatively short lengths of beams (0.40 m or 0.45 m long) between the junctions to (a) reduce propagation losses, (b) increase the likelihood of indirect coupling and (c) allow measurable velocity levels on the furthest beam without using excessive levels of vibration. The wave theory derivations in Section 2 were used to check that the nearfield generated at one junction was negligible in comparison to the propagating wave vibration when it reached the next junction. The nearfield was calculated to be at least 32 dB below the propagating wave at 125 Hz, and 228 dB below the propagating wave at 16 kHz.

The beams had a cross-section of 0.02 m x 0.01 m and were made from solid Perspex for which material property measurements gave the Young’s modulus (mean, \(\mu = 4.59 \times 10^9\) N/m² and standard deviation,
\[ \sigma = 0.51 \times 10^9 \text{N/m}^2, \] density \((1184 \text{ kg/m}^3)\) and the frequency-dependent internal loss factor (see [1]). Poisson’s ratio was assumed to be 0.3. The grillage was built from six beams to minimise the number of beams that needed to be connected with cyanoacrylate glue. The grillage was resiliently suspended in mid-air using elastic (see Fig. 6).

### 3.2 Measurement set-up

For each direction of excitation corresponding to the BL or the BT model, four excitation positions and four measurement positions were used on each beam as shown in Figure 5. Bending velocities were measured on all beams except beams 2, 5 and 8 due to symmetry. Measurement of longitudinal or torsional motion was not attempted as it was prone to error in the presence of bending motion. Point excitation using broadband noise was applied to the source beam using an electrodynamic shaker (Brüel & Kjær Type 4810) with a metal stud connection. A force transducer was not used because (a) the analysis was based on the assessment of energy level differences and (b) the mass of the force transducer could have significantly affected the response of the lightweight Perspex beams. The response was averaged from excitation at a number of different points for comparison with FEM, SEA and ASEA. Analysis was carried out using a Brüel & Kjær PULSE analyser (Nærum, Denmark) in one-third octave bands from 10 Hz to 20 kHz, which were then combined to give octave bands. On all beams the measured levels were at least 10 dB above the background level; however, this required use of a graphic equalizer to boost the signal level above 8 kHz. To avoid mass loading from accelerometers, a laser vibrometer (Polytec PDV100, Waldbronn, Germany) was used to measure the velocity of each beam that was associated with bending wave motion.

### 4 Finite element modeling

Finite element calculations were carried out using Abaqus/CAE 6.12 with mode-based steady state dynamic analysis. Octave band values between 125 Hz and 16 kHz were determined by averaging results from 42 frequencies that were equally spaced over each band. Abaqus beam element B33 was used for the Euler-Bernoulli model and B31 for the Timoshenko beam model. The continuum solid element C3D8 was also used for comparison with measurements. Over the frequency range of interest, the element size was less than \(\frac{1}{10}\) of the bending wavelength (Types A and B) for B33 and B31, and less than \(\frac{1}{20}\) for C3D8.

Rain-on-the-roof excitation was applied to the source beam. For excitation of bending or longitudinal waves, unity forces with random phase were applied at each unconstrained node along the length of beam 1 (approximately 90 nodes). For excitation of torsional waves, unity moments (about the axis running along the beam length) with random phase were applied. Ten sets of rain-on-the-roof with different random phase were used and the results were processed to calculate a mean value with 95% confidence intervals using Student’s \(t\)-distribution. The resulting 95% confidence intervals were typically between 0.5dB and 2dB which was...
sufficient to make comparisons between the different SEA and ASEA models, and these confidence intervals were generally smaller than those from the measurements.

5 Results and discussion

The modal properties of the beams are discussed in Section 5.1 followed by transmission coefficients in Section 5.2. Section 5.3 contains a comparison of FEM, SEA and ASEA for isolated T-junctions to confirm the validity of the wave theory for the BL and BT models. For these T-junctions, Section 5.4 then compares FEM models using three different elements with measurements to establish their validity. Sections 5.5 and 5.6 (for the BL and BT models respectively) compare results from FEM, SEA and ASEA on the three-bay linear grillage. Note that a reduced set of comparisons using only beams 1, 3, 7, 9 and 10 is considered due to the symmetry of the structure. Each graph that shows the energy level difference between two subsystems starts at the octave band frequency which contains the highest fundamental mode from these two subsystems. Comparisons of FEM, SEA and ASEA are also carried out for longitudinal or torsional rain-on-the-roof excitation because in comparison to bending waves, the propagation losses for longitudinal or torsional waves tend to be lower and therefore it is of interest to assess the strength of indirect coupling in these cases. Finally, Section 5.7 assesses the accuracy of ASEA for the three-bay linear grillage when it is considered as perfectly periodic, and when it is imperfectly periodic by introducing uncertainty into the Young’s modulus of the beams.

5.1 Modal properties

Local mode counts of the isolated beams (assuming clamped ends) are shown in Figure 7. The fundamental bending mode (Type A or B) occurs in the 125 Hz or 250 Hz octave band. In previous work [1] it was concluded that when consecutive frequency bands that are above the fundamental mode frequency have at least one local mode on the source or receiving beams, the modal fluctuations are significantly reduced and that this facilitates comparison with SEA and ASEA. Hence using octave bands in this paper satisfies this requirement above 125 Hz and 250 Hz for Type A and B bending waves respectively.

Modal overlap factors are shown in Figure 8 for bending, longitudinal and torsional modes of the beams in the three-bay linear grillage. These are calculated from the total loss factor (sum of the internal loss factor and all the coupling loss factors) used in the SEA and ASEA models and the statistical modal density (which results in smooth curves). A lower and upper limit is calculated for each wave type because the beams have different modal densities and different total loss factors. The modal overlap factor is less than unity below the 16 kHz octave band for Type A and B bending waves, longitudinal waves and torsional waves.

5.2 Transmission coefficients

The three-bay linear grillage of beams is formed from L- and T-junctions. Transmission coefficients for the L-junctions that form part of the three-bay linear grillage are given in Section IV.B.3 in reference [1]. Note that these were one-third octave band values calculated at the band centre frequency; hence, the octave band value corresponds to the one-third octave band with the same centre frequency. Transmission coefficients for the BL and BT models of the T-junctions are shown in Figures 9 and 10. In all cases, the sum of the transmission and reflection coefficients equals unity which satisfies the requirement for conservation of energy. For all T-junctions, transmission around the corner from bending waves on beam 1 to bending waves on beam 2 ($t_{1B2}$) is approximately constant with frequency. Moore [18] indicated that transmission coefficients from bending to torsional waves were smallest; however, transmission coefficients from bending to torsional (and bending to longitudinal) tend to increase with increasing frequency such that they often become larger than those from bending to bending at high frequencies.

For the BL model, around the corner of the T123-junction from beam 1 to 2, the transmission coefficients for $t_{B1L2}$ and $t_{T1B2}$ tend to increase with increasing frequency (Fig. 9a). At low-frequencies there is very high transmission of longitudinal waves from beam 1 to 3 ($t_{L1L3} > 0.8$) which slowly decreases with increasing frequency. For the T124-junction, both the transmission around the corner from bending waves on beam 1 to longitudinal waves on beam 2 ($t_{B1L2}$) and the transmission around the corner from longitudinal waves on beam 1 to bending waves on beam 2 ($t_{L1B2}$) tend to increase with increasing frequency (Fig. 9b).

For the BT model, the transmission coefficient, $t_{B1B2}$, across the straight section of the T123-junction (Fig. 10a) is significantly higher than the others. For the BT model of the T123-junction and T124-junction (Fig. 10b), there are non-zero transmission coefficients around the corner which involve wave conversion (i.e. $t_{B1T2}$ and $t_{T1B2}$) and these tend to increase with increasing frequency.
To assess the validity of the wave theory for the isolated T123- and T124-junctions, energy level differences from FEM (Euler-Bernoulli and Timoshenko elements) are compared with SEA and ASEA in Figures 11 and 12 for the BL and BT models respectively. In the lowest frequency band which contains the fundamental bending mode (125 Hz or 250 Hz) there are differences between FEM and ASEA of up to 12 dB at 125 Hz (e.g. see Figs. 11a and 11e); this is attributed to the low mode count. However, between the band containing the fundamental mode (bending, longitudinal or torsional) and 4 kHz, Figures 11a, 11b, 11d, 11e and 12a, 12b, 12d, 12e show that FEM using Euler-Bernoulli and Timoshenko elements are similar, and are close to ASEA predictions (≤1 dB difference). In the 8 kHz and 16 kHz bands in Figures 11a, 11e and 12b the 95% confidence intervals for FEM with Euler-Bernoulli elements only overlap ASEA with Euler-Bernoulli group velocity, and for FEM with Timoshenko elements they only overlap ASEA with Timoshenko group velocity.

The fact that SEA and ASEA using Euler-Bernoulli theory are similar (≤1.3 dB difference), and SEA and ASEA using Timoshenko theory are similar (≤1.2 dB difference) indicates that there is no significant indirect coupling for these isolated T-junctions. Previous work [1] indicated that the changeover between Euler-Bernoulli and Timoshenko theories is appropriate above the frequency where there is a 26% difference in their group velocities; this occurs in the 16 kHz octave band for Type A bending waves and the 8kHz octave band for Type B bending waves. This results in differences between ASEA using Timoshenko and Euler-Bernoulli group velocities up to 1.6 dB in the 16 kHz band as indicated in Figures 11a, 11b, 11e and 12a, 12b, 12e.

Figures 11c, 11f and 12c, 12f show energy level differences between subsystems for which the transmission coefficients between these subsystems are zero. FEM gives energy level differences over 150 dB; these are sufficiently

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**Figure 7.** Local mode counts on beams 1 and 2: (a) Type A bending, (b) Type B bending, (c) Longitudinal, (d) Torsional.

**Figure 8.** Modal overlap factor for the different wave types: (a) BL model, (b) BT model.

### 5.3 T-junctions: Comparison of FEM, SEA and ASEA

To assess the validity of the wave theory for the isolated T123- and T124-junctions, energy level differences from FEM (Euler-Bernoulli and Timoshenko elements) are...
Figure 9. T-junction – wave theory transmission coefficients for the BL model: (a) T123-junction, (b) T124-junction.

Figure 10. T-junction – wave theory transmission coefficients for the BT model: (a) T123-junction, (b) T124-junction.
The reason that SEA and ASEA predict significantly lower level differences (<30 dB) than FEM is that there are flanking transmission paths between these subsystems. For example, for $E_{L1}/E_{L2}$ (T123-junction on Fig. 11c) $r_{L1L2} = 0$ but energy flows along other transmission paths involving non-zero transmission coefficients such as $L1 \rightarrow B1 \rightarrow L2$ and $L1 \rightarrow B3 \rightarrow L2$. The influence of these flanking paths is not apparent in the FEM response because beams 1 and 3 are identical in their modal response when they have identical dimensions and material properties. However, in Section 5.7, it will be shown that when uncertainty is introduced into the Young’s modulus of each beam, FEM will then become similar to ASEA for subsystems where the transmission coefficients between them are zero.
The BL model for the isolated T123-junction allows an assessment of whether there are any issues due to high transmission coefficients when there is excitation of longitudinal waves on L1 (source subsystem) when the receiving subsystem is L3. Referring back to Figure 9a, $\tau_{11L3} > 0.7$ above 2 kHz. With FEM these high levels of transmission are partly confirmed by level differences of $\approx 0$ dB between 2 kHz and 8 kHz (see Fig. 11d). However, whilst SEA is consistently higher than FEM by $\approx 2.7$ dB there is reasonable agreement between FEM and ASEA (within 1.5 dB) because the propagating energy transfers that are modelled with ASEA can incorporate such high transmission coefficients.

The results confirm the validity of the wave theory developed for the BL and BT models and this theory is now used to model the junctions that form the grillage.
5.4 Grillage: Comparison of FEM models using different element types with measurements

For excitation of Type A and B bending waves on beam B1 of the grillage, Figures 13 and 14 compare measurements with FEM models using Euler-Bernoulli, Timoshenko and continuum solid elements. The latter element is included to see whether it shows closer agreement with measurements than Euler-Bernoulli and Timoshenko elements.

Note that Euler-Bernoulli and Timoshenko theories have \( \geq 26\% \) difference in their group velocities in the 16 kHz and 8 kHz octave bands for Type A and Type B bending waves respectively.

The first step is to assess \( E_{B1}/E_{B3} \) for adjacent coupled beams because this avoids any accumulation of errors involving wave conversion across more than one junction. For Type A bending waves (Fig. 13a) the 95\% confidence intervals from measurements overlap with Euler-Bernoulli,
Timoshenko and continuum solid elements between 125 Hz and 4 kHz but not with Timoshenko elements at 8 kHz and 16 kHz. For Type B bending waves (Fig. 14a), FEM results from Euler-Bernoulli, Timoshenko and continuum solid elements are within 1 dB of each other, and the 95% confidence intervals tend to overlap with measurements. The next step is to consider the non-adjacent beams after significant wave conversion has occurred at the junctions.

For Type A bending waves (Figs. 13b–13f) the general conclusion is that all three finite elements show reasonable agreement with measurements up to 16 kHz. For Type B bending waves, there is evidence that Euler-Bernoulli theory is no longer appropriate above 2 kHz (Figs. 14c–14f), but both Timoshenko and continuum solid elements show reasonable agreement with measurements. In general, the agreement between measurements and FEM using Euler-Bernoulli or Timoshenko elements indicates that it is sufficient to consider only these two theories in SEA and ASEA models, as well as in the FEM models in subsequent sections.

Figure 14. Three-bay linear grillage – comparison of measurements and FEM (Euler-Bernoulli (E-B), Timoshenko (T) and Continuum Solid (SC)). Excitation of Type B bending waves on subsystem B1.
5.5 Grillage: BL model

For the BL model, this section compares FEM, SEA and ASEA. Figure 15 compares measured and predicted energy level differences for the BL model where Type A bending waves (Figs. 15a–15d) or longitudinal waves (Figs. 15e and 15f) are excited on beam 1. For octave bands from 125 Hz to 1 kHz there are only bending modes, and each band typically contains at least one local bending mode for the source and receiving beams. For $E_{B1}/E_{B3}$ (i.e. adjacent coupled beams) SEA and ASEA are nominally identical (see Fig. 15a) and there is close agreement (within 3 dB) between FEM, SEA and ASEA (Euler-Bernoulli and Timoshenko theories). ASEA gives...
lower level differences than SEA as the source and receiving subsystems become more distant from each other, for example see $E_{B1}/E_{B10}$ (i.e. transmission to the most distant beam) in Figure 15b; this indicates the existence of significant indirect coupling between non-adjacent beams. However, there are differences between ASEA and FEM where $E_{B1}/E_{B10}$ from ASEA (Euler-Bernoulli and Timoshenko group velocity) is $\approx 3$ dB higher than FEM (Euler-Bernoulli and Timoshenko elements).

Between 2 kHz and 16 kHz there are both bending and longitudinal modes, and measurements show closer agreement with FEM using Euler-Bernoulli elements than Timoshenko elements for $E_{B1}/E_{B3}$ (also $E_{B1}/E_{B6}$ and $E_{B1}/E_{B9}$ which, for brevity, are not included here).
However, measurements and FEM using Euler-Bernoulli or Timoshenko elements are similar for \( E_{B1}/E_{B10} \) (also for \( E_{B1}/E_{B4} \) and \( E_{B1}/E_{B7} \) which, for brevity, are not included here). In general, FEM using Euler-Bernoulli elements shows closest agreement with ASEA using Euler-Bernoulli group velocity, and FEM using Timoshenko elements shows closest agreement with ASEA using Timoshenko group velocity. This is evident in the 8 kHz and 16 kHz bands for \( E_{B1}/E_{B6} \) and \( E_{B1}/E_{B9} \) than for \( E_{B1}/E_{B4} \), \( E_{B1}/E_{B7} \), and \( E_{B1}/E_{B10} \). With increasing frequency, the generation of longitudinal waves at the junction typically increases the indirect coupling such that ASEA gives significantly lower energy level differences than SEA as the beams become more distant from the source. Due to the existence of indirect coupling, measurements and FEM generally show closer agreement with ASEA rather than SEA.

Figure 15f allows comparison of \( E_{L1}/E_{L10} \) for the BL model where longitudinal waves are excited on the source subsystem and the receiving subsystem represents longitudinal wave energy. For the T123 junction, \( \tau_{L1L3} > 0.7 \) above 2 kHz; hence there will be high transmission of longitudinal wave energy between subsystems L3 and L6, and L6 and L9. Note that there is reasonable agreement (<4 dB) between FEM and ASEA using both Euler-Bernoulli and Timoshenko theory for \( E_{L1}/E_{L3} \), \( E_{L1}/E_{L6} \) and \( E_{L1}/E_{L9} \) (which, for brevity, are not included here). This supports the finding from the isolated T-junctions (Sect. 5.3) that ASEA remains valid when it incorporates high transmission coefficients. However, for \( E_{L1}/E_{L10} \) (also for \( E_{L1}/E_{L4} \), \( E_{L1}/E_{L7} \) which, for brevity, are not included here) there are significant differences between FEM and ASEA and these differences become larger as the receiving subsystem becomes more distant from the source subsystem leading...
to a difference of \( \approx 11 \) dB for \( E_{B1}/E_{B10} \). This is due to the “perfectly periodic” grillage; hence, the effect of uncertainty in the material properties will be examined in Section 5.7.

5.6 Grillage: BT model

For the BT model, this section compares FEM, SEA and ASEA. Figure 16 compares measured and predicted energy level differences for the BT model where Type B bending waves (Figs. 16a–16d) or torsional waves (Figs. 16e and 16f) are excited on beam 1.

The two octave bands from 250 Hz to 500 Hz contain only bending modes and each band contains at least one local bending mode for the source and receiving beams. For adjacent coupled beams, \( E_{B1}/E_{B10} \) in Figure 16a shows close agreement (within \( \approx 3 \) dB) between measurements, FEM, SEA and ASEA (Euler-Bernoulli and Timoshenko theory are nominally identical). However, for \( E_{B1}/E_{B10} \) in Figure 16b, SEA overestimates the measured energy level difference by \( \approx 6 \) dB and there is closer agreement (\( \leq 3.2 \) dB) between measurements, FEM and ASEA. The existence of indirect coupling is indicated by ASEA having lower energy level differences than SEA.

Between 1 kHz and 16 kHz there are both bending and torsional modes. Measurements tend to show closest agreement with FEM using Timoshenko theory. The difference between these FEM models becomes apparent at and above 4 kHz although the difference between Timoshenko and Euler-Bernoulli group velocities is only \( \geq 26\% \), at and above the 8 kHz band. In general, ASEA with Timoshenko group velocity shows closest agreement with measurements and FEM using Timoshenko theory.

Above 2 kHz for \( E_{B1}/E_{B10} \) (Fig. 16b) there is no longer close agreement between FEM using Euler-Bernoulli elements and ASEA using Euler-Bernoulli group velocity as was previously observed with the furthest beam in a rectangular beam frame in reference [1]. For practical purposes, this is not significant because Timoshenko theory will usually be chosen, but it indicates that there may also be issues due to the perfectly periodic structure; this is considered further in Section 5.7.

Figures 16c and 16d allow comparison of predicted energy level differences where Type B bending waves are excited on the source subsystem and the receiving subsystem represents torsional wave energy. For \( E_{B1}/E_{T3} \)
ASEA and FEM using Euler-Bernoulli and Timoshenko theory are similar. However, for $E_{B1}/E_{T10}$ (Fig. 16d), FEM using Euler-Bernoulli elements show closest agreement with ASEA using Euler-Bernoulli group velocity, and FEM using Timoshenko elements shows closest agreement with ASEA using Timoshenko group velocity.

Figure 16e allows comparison of $E_{T1}/E_{B10}$ where torsional waves are excited on the source subsystem and the receiving subsystem represents Type B bending wave energy. ASEA (Euler-Bernoulli or Timoshenko) and FEM (Euler-Bernoulli or Timoshenko) show the same trends above 4 kHz with a maximum difference of ≈2 dB.

Figure 16f allows comparison of $E_{T1}/E_{T10}$ where torsional waves are excited on the source subsystem and the receiving subsystem represents torsional wave energy. As noted above, the difference between SEA and ASEA indicates significant indirect coupling. Above 4 kHz there is close agreement between FEM and ASEA when they both use either Euler-Bernoulli or Timoshenko theory.

5.7 Grillage: Effect of uncertainty in the Young’s modulus for BL and BT models

In previous sections, the modelling has considered a perfectly periodic grillage with uniform material properties. In order to assess the effect of uncertainty in the material properties, a Monte Carlo simulation is now carried out using FEM. An ensemble of ten different models of the grillage is created in which the Young’s modulus for each beam is determined by random sampling of values from a normal distribution based upon the mean and standard deviation from the measured Young’s modulus for Perspex. For these data, the coefficient of variation ($\sigma/\mu$) is 0.11.

For the BL model in Section 5.5, it was noted that with longitudinal excitation there were a few energy level differences with discrepancies up to 11 dB between ASEA and FEM for the perfectly periodic grillage. For $E_{L1}/E_{B10}$, $E_{L1}/E_{B10}$, $E_{L1}/E_{L7}$ and $E_{L1}/E_{L10}$. Figure 17 shows a comparison of ASEA with FEM using uniform material properties and FEM with random Young’s modulus. This confirms that there is significantly closer agreement (typically within 3 dB) between ASEA and FEM with the

![Grillage Diagram]

**Figure 19.** Three-bay linear grillage – Difference between FEM and ASEA energy level differences for BL model (source subsystem: L1): (a) FEM (uniform material properties) with Euler-Bernoulli elements; (b) FEM (random Young’s modulus) with Euler-Bernoulli elements; (c) FEM (uniform material properties) with Timoshenko elements; (d) FEM (random Young’s modulus) with Timoshenko elements. -- bending wave energy receiving subsystem; --- longitudinal wave energy receiving subsystem.
random Young’s modulus when the local modes of the beams that form the grillage are no longer identical.

For the BL and BT models, Figures 18–21 show the differences between the energy level differences predicted using FEM and those from ASEA. In general, the differences are largest in the first few frequency bands and show the largest fluctuations. This supports the finding from [1] that only when each beam supports at least two local modes for each wave type that occurs in the frequency band of interest, does the response tend to be sufficiently smooth that reasonable agreement occurs with ASEA. However, across the entire frequency range the results indicate that regardless of whether Euler-Bernoulli or Timoshenko theory is used, the effect of uncertainty in the Young’s modulus is to reduce the differences between FEM and ASEA to less than ≈3 dB. This effect is most pronounced with excitation of longitudinal waves rather than excitation of bending or torsional waves. ASEA with multiple wave types is able to predict the response on the “perfectly periodic” structure with bending wave excitation (Figs. 18a and 18c) typically to within ≈3 dB (BL model) or ≈4 dB (BT model). This is noteworthy because this level of accuracy is similar to that achieved with ASEA considering only bending waves on a periodic ribbed plate [20] and a repeating box-like structure formed from isotropic homogeneous plates [21]. Hence there is a growing body of evidence to suggest that when the component subsystems support at least a few local modes, ASEA can be used to model real-world engineering structures, particularly because they tend to be imperfectly periodic to varying degrees.

6 Conclusions

A three-bay linear grillage has been created from relatively short beams to ensure that tunneling mechanisms were more important than propagation losses over the majority of the frequency range. The importance of indirect coupling was confirmed for both bending-longitudinal and bending-torsional models because (a) ASEA predicted higher energy levels than SEA on beams that were distant from the source and (b) there was closer agreement between FEM and ASEA (rather than SEA) at low-frequencies where there were only bending modes on all the beams,
and at high frequencies where the beams supported longitudinal or torsional modes as well as bending modes.

When the difference between Timoshenko and Euler-Bernoulli group velocities was ≥26%, there were significant differences between FEM models using Euler-Bernoulli and Timoshenko elements as well as SEA and ASEA models using Euler-Bernoulli and Timoshenko group velocities. However, the FEM models showed closest agreement with ASEA rather than SEA. This validates the approach proposed in [1] to incorporate Timoshenko theory into ASEA purely by changing the group velocity used to calculate the coupling loss factors. Measurements using bending wave excitation showed closest agreement with FEM using Euler-Bernoulli elements when longitudinal waves were generated at the junction, but in this paper there was stronger evidence of close agreement between measurements and FEM using Timoshenko elements when torsional waves were generated at the junction.

For bending wave excitation on a “perfectly periodic” finite grillage where the component beams have identical material properties there was agreement between measurements, FEM and ASEA. The differences between FEM and ASEA could be partly attributed to neglecting phase effects in ASEA, but these were less than ≈5 dB. However, with longitudinal wave excitation there were some beam subsystems that did not show such close agreement between FEM and ASEA. Hence, numerical experiments with FEM were carried out to introduce uncertainty into the Young’s modulus associated with each beam. This showed that for beams modelled with either Euler-Bernoulli or Timoshenko theory, the effect of this uncertainty is to reduce the differences between FEM and ASEA to less than ≈3 dB.

The results demonstrate that ASEA can be used with multiple wave types to predict vibration transmission across perfectly or imperfectly periodic frameworks of beams at frequencies where the component beams support local modes. The existence and importance of indirect coupling can be identified by the difference between SEA and ASEA predictions.

**Conflict of interest**

The authors declare that they have no conflicts of interest in relation to this article.
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References


Appendix A

A Derivations for wave transmission coefficients of T-junctions

A.1 BL model for T123-junction

Consider an incident Type A bending wave travelling in the positive x-direction towards the junction on beam 1 (see Fig. 2a). Both bending (Type A) and longitudinal waves will be transmitted to beam 2. Due to symmetry, longitudinal waves in beams 1 and 3 have the same magnitude but travel in opposite directions. The incident wave is assumed to have unit amplitude and the wave fields are given by:

\[ v_{B1} = \left( e^{-ik_{B1}x} + r_{B1B2} e^{ik_{B1}x} + r_{B1B3} e^{ik_{B1}x} \right) e^{i\omega t}, \]  

\[ v_{B2} = \left( t_{B1B2} e^{ik_{B2}x} + t_{N2} e^{-ik_{B2}x} \right) e^{i\omega t}, \]  

\[ v_{B3} = \left( t_{B1B3} e^{-ik_{B3}x} + t_{N3} e^{ik_{B3}x} \right) e^{i\omega t}, \]  

where subscripts B, L, and N represent bending, longitudinal and nearfield respectively, \( r \) indicates the complex amplitude of the reflected wave, \( t \) indicates the complex amplitude of the transmitted wave.

Continuity of velocity in both \( x \) and \( y \) direction at the junction requires that

\[ v_{B1} = v_{L1} = v_{N1} = v_{B2}, \]  

Continuity of angular velocity at the joint requires that

\[ \frac{\partial v_{B1}}{\partial x} = \frac{\partial v_{B2}}{\partial y}, \quad \frac{\partial v_{B1}}{\partial x} = \frac{\partial v_{B3}}{\partial x}. \]  

The force equilibrium relationships in the \( x \) and \( y \)-directions are given by

\[ F_{L1} + F_{B2} + F_{L3} = 0, \]  

\[ F_{B1} - F_{L2} - F_{B3} = 0. \]

Moment equilibrium for bending motion on the three beams is described by

\[ M_{B1} - M_{B2} - M_{B3} = 0. \]
The calculations of forces and moments are given by

\[ F_B = EI \int \frac{\partial^2 v_B}{\partial x^2} \, dx, \quad \text{(A12)} \]

\[ F_L = -\rho A \int \frac{\partial v_L}{\partial t} \, dx, \quad \text{(A13)} \]

\[ M_B = -EI \int \frac{\partial^2 v_B}{\partial x^2} \, dx, \quad \text{(A14)} \]

\[ M_T = -\rho J \int \frac{\partial \omega_T}{\partial t} \, dx, \quad \text{(A15)} \]

where \( J \) is the polar moment of inertia of the cross-section.

Equations (A7)–(A11) give eight boundary conditions that result in matrix equation (1).

Now consider an incident longitudinal wave travelling in the positive \( x \)-direction towards the junction on beam 1 (see Fig. 2b). This will lead to a Type A bending wave on beams 1, 2 and 3, and a longitudinal wave reflected onto beam 1, and a longitudinal wave transmitted to beam 3. Due to structural symmetry, the bending waves on beams 1 and 3 have the same magnitudes but travel in opposite directions with a phase difference of \( \pi \) between them. This causes zero displacement in the \( y \)-direction at the junction. Thus, in beam 2 there is only bending wave motion. The incident wave is assumed to have unit amplitude; hence the wave fields are described as follows:

\[ v_{L1} = (e^{-ik_{11}x} + r_{L11}e^{ik_{11}x}) e^{i\omega t}, \quad \text{(A16)} \]

\[ v_{L2} = 0, \quad \text{(A17)} \]

\[ v_{L3} = t_{L1L3} e^{-ik_{11}x} e^{i\omega t}, \quad \text{(A18)} \]

\[ v_{B1} = (t_{L1B1} e^{ik_{21}x} + t_{N1} e^{ik_{11}x}) e^{i\omega t}, \quad \text{(A19)} \]

\[ v_{B2} = (t_{L1B2} e^{-ik_{12}y} + t_{N2} e^{-ik_{21}y}) e^{i\omega t}, \quad \text{(A20)} \]

\[ v_{B3} = -(t_{L1B3} e^{-ik_{11}x} + t_{N3} e^{-ik_{21}x}) e^{i\omega t}. \quad \text{(A21)} \]

Continuity of angular velocity at the joint requires that

\[ \frac{\partial v_{B1}}{\partial x} = \frac{\partial v_{B2}}{\partial y}, \quad \text{(A22)} \]

and continuity of translational velocity in \( x \) and \( y \)-directions requires that

\[ v_{L1} = v_{B2} = v_{L3}, \quad v_{B1} = -v_{B3} = 0. \quad \text{(A23)} \]

In the \( y \)-direction, \( F_{B1} - F_{B3} = 0 \), and in the \( x \)-direction, force equilibrium requires that

\[ F_{L1} - F_{B1} - F_{L3} = 0. \quad \text{(A24)} \]

At the junction, \( M_{B1} + M_{B3} = 0 \), and moment equilibrium gives

\[ 2M_{B1} - M_{B2} = 0. \quad \text{(A25)} \]

Equations (A22)–(A25) give six boundary conditions that result in matrix equation (8).

**A.2 BL model for T124-junction**

Consider an incident Type A bending wave with unit amplitude travelling in the positive \( x \)-direction towards the junction on beam 1 (see Fig. 3a). Both bending (Type A) and longitudinal waves will be transmitted onto beams 2 and 4. Due to symmetry, bending waves on beams 2 and 4 have the same magnitude but travel in opposite directions. Hence there will be zero displacement in the \( x \)-direction at the junction. Similarly, the phases are different but magnitudes are the same for longitudinal waves on beams 2 and 4. The wave fields of T124-junction for bending wave excitation are described by

\[ v_{B1} = (e^{-ik_{11}x} + r_{B1B1} e^{ik_{21}x} + r_{N1} e^{ik_{11}x}) e^{i\omega t}, \quad \text{(A26)} \]

\[ v_{B2} = (t_{B1B2} e^{-ik_{12}y} + t_{N2} e^{-ik_{21}y}) e^{i\omega t}, \quad \text{(A27)} \]

\[ v_{B4} = -(t_{B1B4} e^{ik_{11}x} + t_{N4} e^{ik_{21}x}) e^{i\omega t}, \quad \text{(A28)} \]

\[ v_{L1} = 0, \quad \text{(A29)} \]

\[ v_{L2} = t_{B1L2} e^{ik_{21}y} e^{i\omega t}, \quad \text{(A30)} \]

\[ v_{L4} = -t_{B1L4} e^{ik_{12}y} e^{i\omega t}. \quad \text{(A31)} \]

Continuity of angular velocity requires that

\[ \frac{\partial v_{B1}}{\partial x} = \frac{\partial v_{B2}}{\partial y}, \quad \text{(A32)} \]

and continuity of velocity in \( x \) and \( y \)-directions gives

\[ v_{B1} = v_{L2}, \quad v_{B2} = -v_{B4} = 0. \quad \text{(A33)} \]

In the \( x \)-direction, \( F_{B2} - F_{B4} = 0 \), and in the \( y \)-direction, \( F_{L2} = F_{L4} \), and force equilibrium requires that

\[ F_{B1} - 2F_{L2} = 0. \quad \text{(A34)} \]

At the junction, \( M_{B2} = -M_{B4} \), and moment equilibrium gives

\[ M_{B1} - 2M_{B2} = 0. \quad \text{(A35)} \]

This gives the matrix equation in equation (14).

Now consider an incident longitudinal wave travelling in the positive \( x \)-direction towards the junction on beam 1 (Fig. 3b). This will generate Type A bending waves on beams 2 and 4, and a longitudinal wave reflected on beam 1. At the junction, the rotational displacement is zero due to balanced bending moments from beams 2 and 4. The incident wave is assumed to have unit amplitude; hence the wave fields are described as follows:

\[ v_{L1} = (e^{-ik_{11}x} + r_{L1L1} e^{ik_{11}x}) e^{i\omega t}, \quad \text{(A36)} \]
At the junction, the velocity is zero in the z-direction. In addition, the angular velocity is zero where
\[
\frac{\partial \theta_{\text{B}2}}{\partial y} = 0. \tag{A42}
\]
In the x-direction, \( F_{\text{B}2} = -F_{\text{B}4} \), and force equilibrium requires that
\[
F_{\text{L}1} - 2F_{\text{B}2} = 0. \tag{A43}
\]
For the bending moment, \( M_{\text{B}2} - M_{\text{B}4} = 0 \); hence these three equations give the matrix in equation (19).

### A.3 BT model for T124-junction

Consider an incident Type B bending wave travelling in the positive x-direction towards the junction on beam 1 (see Fig. 5a). The transmitted bending wave (Type B) on beams 2 and 4 has opposite moments at the junction in the x-direction, hence there is no torsional wave motion on beam 1. This incident wave is assumed to have unit amplitude; hence the wave fields can be described as follows:
\[
\begin{align*}
v_{\text{B}1} &= \left(e^{-ik_{\text{B}1}x} + r_{\text{B}1\text{B}2}e^{ik_{\text{B}1}x} + r_{\text{B}1\text{B}4}e^{ik_{\text{B}1}x}\right)e^{\text{out}}, \\
v_{\text{B}2} &= \left(t_{\text{B}1\text{B}2}e^{-ik_{\text{B}2}y} + t_{\text{B}2\text{B}4}e^{-ik_{\text{B}2}y}\right)e^{\text{out}}, \\
v_{\text{B}4} &= \left(t_{\text{B}1\text{B}4}e^{-ik_{\text{B}2}y} + t_{\text{B}2\text{B}4}e^{-ik_{\text{B}2}y}\right)e^{\text{out}}, \\
o_{\text{T}1} &= 0,
\end{align*}
\]
Continuity of angular velocity in the x-direction, and bending velocity in the z-direction requires that
\[
\begin{align*}
o_{\text{T}2} &= o_{\theta_{\text{B}2}} e^{-ik_{\text{B}2}y} e^{\text{out}} = -ik_{\text{B}2}t_{\text{B}1\text{B}2}e^{ik_{\text{B}1}x} e^{\text{out}}, \\
o_{\text{T}4} &= o_{\theta_{\text{B}4}} e^{ik_{\text{B}2}y} e^{\text{out}} = -ik_{\text{B}2}t_{\text{B}1\text{B}4}e^{ik_{\text{B}1}x} e^{\text{out}}.
\end{align*}
\]
Continuity of bending velocity in the z-direction, and rotational velocity in the y-direction requires that
\[
\begin{align*}
\frac{\partial v_{\text{B}1}}{\partial x} &= \omega_{\text{T}2}, \\
\frac{\partial v_{\text{B}2}}{\partial y} &= \omega_{\text{T}1}, \tag{A51}
\end{align*}
\]
Rotational velocity in the x-direction is zero, hence
\[
\frac{\partial v_{\text{B}2}}{\partial y} = 0 = \omega_{\text{T}1}. \tag{A52}
\]
Shear force and moment equilibrium requires that
\[
\begin{align*}
F_{\text{B}1} &= 2F_{\text{B}2}, \tag{A53} \\
M_{\text{B}1} &= 2M_{\text{T}2}. \tag{A54}
\end{align*}
\]
Equations (A50)–(A54) give five boundary conditions which result in the matrix in equation (24).

Now consider an incident torsional wave travelling in the positive x-direction towards the junction on beam 1 (see Fig. 5b). Bending waves (Type B) are transmitted to beams 2 and 4. Due to structural symmetry, the shear forces from bending motion on beams 2 and 4 are balanced and bending displacement is zero in the z-direction at the junction. The incident wave is assumed to have unit amplitude; hence the wave fields can be described as follows:
\[
\begin{align*}
\omega_{\text{T}1} &= \left(e^{-ik_{\text{T}1}x} + r_{\text{T}1\text{T}2}e^{ik_{\text{T}1}x}\right)e^{\text{out}}, \\
\omega_{\text{T}2} &= \omega_{\text{T}4} = 0, \tag{A55} \\
v_{\text{B}1} &= 0, \tag{A56} \\
v_{\text{B}2} &= -\frac{i}{k_{\text{T}2}} \left(t_{\text{T}1\text{T}2}e^{-ik_{\text{T}2}y} + t_{\text{T}2\text{T}4}e^{-ik_{\text{T}2}y}\right)e^{\text{out}}, \tag{A57} \\
v_{\text{B}4} &= \frac{i}{k_{\text{T}2}} \left(t_{\text{T}1\text{T}4}e^{ik_{\text{T}2}y} + t_{\text{T}2\text{T}4}e^{ik_{\text{T}2}y}\right)e^{\text{out}}. \tag{A58}
\end{align*}
\]
Continuity of angular velocity in the x-direction, and bending velocity in the z-direction requires that
\[
\frac{\partial v_{\text{B}2}}{\partial y} = \frac{\partial \theta_{\text{B}2}}{\partial y}, \tag{A59}
\]
At the junction, \( M_{\text{B}2} = M_{\text{B}4} \), and the moment equilibrium relationship in the x-direction gives
\[
M_{\text{T}1} - M_{\text{B}2} + M_{\text{B}4} = M_{\text{T}1} - 2M_{\text{B}2} = 0. \tag{A60}
\]
The three boundary conditions define the matrix equation, equation (30).